

NATURAL CONNECTIONS WITH TORSION EXPRESSED BY THE METRIC TENSORS ON ALMOST CONTACT MANIFOLDS WITH B-METRIC

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ABSTRACT. On a main class of the almost contact manifolds with B-metric, it is described the family of the linear connections preserving the manifold's structures by 4 parameters. In this family there are determined the canonical-type connection and the connection with zero parameters.

INTRODUCTION

The investigations in the differential geometry of the almost contact manifolds with B-metric is initiated in [2]. These manifolds are the odd-dimensional extension of the almost complex manifolds with Norden metric and the case of indefinite metrics corresponding to the almost contact metric manifolds. The geometry of the considered manifolds is the geometry of the both of the structures — the almost contact structure and the B-metric. There are important the linear connections with respect to which the structures are parallel. In the general case such connections, which is called natural connections, are a countless number. There are interesting those natural connections, which torsion tensor is expressed by the metric tensors and the structural 1-forms of the investigated manifold. In our case, this condition restricts the manifolds to the class of the considered manifolds having a conformally equivalent structure of the parallel almost contact B-metric structure with respect to the Levi-Civita connection.

Such a problem on Riemannian product manifolds is studied in [5] and [6].

The present paper¹ is organized as follows. In Sec. 1 we give some necessary facts about the considered manifolds. In Sec. 2 we express the set of the torsion

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tensors of the natural connections in terms of the metric tensors on an almost contact manifold with B-metric as a 4-parametric family. In Sec. 3 we determine the φ -canonical connection and the natural connection with zero parameters in the mentioned family.

1. PRELIMINARIES

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact manifold with B-metric or an *almost contact B-metric manifold*, i.e. M is a $(2n+1)$ -dimensional differentiable manifold with an almost contact structure (φ, ξ, η) consisting of an endomorphism φ of the tangent bundle, a vector field ξ , its dual 1-form η as well as M is equipped with a pseudo-Riemannian metric g of signature $(n, n+1)$, such that the following algebraic relations are satisfied

$$(1.1) \quad \begin{aligned} \varphi\xi &= 0, \quad \varphi^2 = -\text{Id} + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \\ g(x, y) &= -g(\varphi x, \varphi y) + \eta(x)\eta(y) \end{aligned}$$

for arbitrary x, y of the algebra $\mathfrak{X}(M)$ on the smooth vector fields on M .

Further, x, y, z will stand for arbitrary elements of $\mathfrak{X}(M)$.

The associated metric \tilde{g} of g on M is defined by

$$(1.2) \quad \tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y).$$

Both metrics g and \tilde{g} are necessarily of signature $(n, n+1)$. The manifold $(M, \varphi, \xi, \eta, \tilde{g})$ is also an almost contact B-metric manifold.

The covariant derivatives of φ, ξ, η with respect to the Levi-Civita connection ∇ play a fundamental role in the differential geometry on the almost contact manifolds. The structural tensor F of type $(0,3)$ on $(M, \varphi, \xi, \eta, g)$ is defined by $F(x, y, z) = g((\nabla_x \varphi)y, z)$. It has the following properties:

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).$$

The relations of $\nabla\xi$ and $\nabla\eta$ with F are:

$$(\nabla_x \eta)y = g(\nabla_x \xi, y) = F(x, \varphi y, \xi).$$

The following 1-forms are associated with F :

$$\theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z),$$

where g^{ij} are the components of the inverse matrix of g with respect to a basis $\{e_i; \xi\}$ ($i = 1, 2, \dots, 2n$) of the tangent space $T_p M$ of M at an arbitrary point $p \in M$. Obviously, the equality $\omega(\xi) = 0$ and the relation $\theta^* \circ \varphi = -\theta \circ \varphi^2$ are always valid.

A classification of the almost contact manifolds with B-metric with respect to F is given in [2]. This classification includes eleven basic classes $\mathcal{F}_1, \mathcal{F}_2, \dots$,

\mathcal{F}_{11} . Their intersection is the special class \mathcal{F}_0 determined by the condition $F(x, y, z) = 0$. Hence \mathcal{F}_0 is the class of almost contact B-metric manifolds with ∇ -parallel structures, i.e. $\nabla\varphi = \nabla\xi = \nabla\eta = \nabla g = \nabla\tilde{g} = 0$.

In the present paper we consider the manifolds from the so-called main classes \mathcal{F}_1 , \mathcal{F}_4 , \mathcal{F}_5 and \mathcal{F}_{11} . These classes are the only classes where the tensor F is expressed by the metrics g and \tilde{g} . They are defined as follows:

$$\begin{aligned}
 (1.3) \quad \mathcal{F}_1 : \quad F(x, y, z) &= \frac{1}{2n} \{g(x, \varphi y)\theta(\varphi z) + g(\varphi x, \varphi y)\theta(\varphi^2 z)\}_{(y \leftrightarrow z)}; \\
 \mathcal{F}_4 : \quad F(x, y, z) &= -\frac{\theta(\xi)}{2n} \{g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y)\}; \\
 \mathcal{F}_5 : \quad F(x, y, z) &= -\frac{\theta^*(\xi)}{2n} \{g(x, \varphi y)\eta(z) + g(x, \varphi z)\eta(y)\}; \\
 \mathcal{F}_{11} : \quad F(x, y, z) &= \eta(x) \{ \eta(y)\omega(z) + \eta(z)\omega(y) \},
 \end{aligned}$$

where (for the sake of brevity) we use the denotation $\{A(x, y, z)\}_{(y \leftrightarrow z)}$ — instead of $\{A(x, y, z) + A(x, z, y)\}$ for any tensor $A(x, y, z)$.

Let us remark that the class $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$, determined by ([10])

$$\begin{aligned}
 (1.4) \quad F(x, y, z) &= -\frac{1}{2n} \{g(\varphi x, \varphi y)\theta(z) + g(x, \varphi y)\theta^*(z) \\
 &\quad - 2n \eta(x)\eta(y)\omega(z)\}_{(y \leftrightarrow z)},
 \end{aligned}$$

is the odd-dimensional analogue of the class \mathcal{W}_1 of the conformal Kähler manifolds of the corresponding almost complex manifold with Norden metric, introduced in [4].

Definition 1.1 ([11]). A linear connection D is called a *natural connection* on the manifold $(M, \varphi, \xi, \eta, g)$ if the almost contact structure (φ, ξ, η) and the B-metric g are parallel with respect to D , i.e. $D\varphi = D\xi = D\eta = Dg = 0$.

As a corollary, the associated metric \tilde{g} is also parallel with respect to a natural connection D on $(M, \varphi, \xi, \eta, g)$.

According to [13], a necessary and sufficient condition a linear connection D to be natural on $(M, \varphi, \xi, \eta, g)$ is $D\varphi = Dg = 0$.

If T is the torsion of D , i.e. $T(x, y) = D_x y - D_y x - [x, y]$, then the corresponding tensor of type (0,3) is determined by $T(x, y, z) = g(T(x, y), z)$.

Let us denote the difference between the natural connection D and the Levi-Civita connection ∇ of g by $Q(x, y) = D_x y - \nabla_x y$ and the corresponding tensor of type (0,3) — by $Q(x, y, z) = g(Q(x, y), z)$.

Proposition 1.1 ([11]). *A linear connection D is a natural connection on an almost contact B-metric manifold if and only if*

$$Q(x, y, \varphi z) - Q(x, \varphi y, z) = F(x, y, z), \quad Q(x, y, z) = -Q(x, z, y).$$

Hence and $T(x, y) = Q(x, y) - Q(y, x)$ we have the equality of Hayden's theorem [7]

$$(1.5) \quad Q(x, y, z) = \frac{1}{2} \{T(x, y, z) - T(y, z, x) + T(z, x, y)\}.$$

Definition 1.2 ([15]). A natural connection D is called a φ -canonical connection on the manifold $(M, \varphi, \xi, \eta, g)$ if the torsion tensor T of D satisfies the following identity:

$$(1.6) \quad \begin{aligned} & \{T(x, y, z) - T(x, \varphi y, \varphi z) - \eta(x) \{T(\xi, y, z) - T(\xi, \varphi y, \varphi z)\} \\ & - \eta(y) \{T(x, \xi, z) - T(x, z, \xi) - \eta(x)T(z, \xi, \xi)\}\}_{[y \leftrightarrow z]} = 0, \end{aligned}$$

where we use the denotation $\{A(x, y, z)\}_{[y \leftrightarrow z]}$ instead of $A(x, y, z) - A(x, z, y)$ for any tensor $A(x, y, z)$.

Let us remark that the restriction the φ -canonical connection D of the manifold $(M, \varphi, \xi, \eta, g)$ on the contact distribution $\ker(\eta)$ is the unique canonical connection of the corresponding almost complex manifold with Norden metric, studied in [3].

In [12], it is introduced a natural connection on $(M, \varphi, \xi, \eta, g)$, defined by

$$(1.7) \quad \nabla_x^0 y = \nabla_x y + Q^0(x, y),$$

where $Q^0(x, y) = \frac{1}{2} \{(\nabla_x \varphi) \varphi y + (\nabla_x \eta) y \cdot \xi\} - \eta(y) \nabla_x \xi$. Therefore, we have

$$Q^0(x, y, z) = \frac{1}{2} \{F(x, \varphi y, z) + \eta(z)F(x, \varphi y, \xi) - 2\eta(y)F(x, \varphi z, \xi)\}.$$

In [14], the connection determined by (1.7) is called a φB -connection. It is studied for some classes of $(M, \varphi, \xi, \eta, g)$ in [12], [8], [9] and [14]. The φB -connection is the odd-dimensional analogue of the B-connection on the corresponding almost complex manifold with Norden metric, studied for the class \mathcal{W}_1 in [1].

In [15], it is proved that the φ -canonical connection and the φB -connection coincide on the almost contact B-metric manifolds in the class $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$. Therefore, these connections coincide also in the class $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$.

The torsion of the φ -canonical connection has the following form ([14])

$$(1.8) \quad T^0(x, y, z) = \frac{1}{2} \{F(x, \varphi y, z) + \eta(z)F(x, \varphi y, \xi) + 2\eta(x)F(y, \varphi z, \xi)\}_{[x \leftrightarrow y]}.$$

In [13], it is given a classification of the linear connections on the almost contact B-metric manifolds with respect to their torsion tensors T in 11 classes \mathcal{T}_{ij} . The characteristic conditions of these basic classes are the following:

$$\begin{aligned}
 (1.9) \quad \mathcal{T}_{11/12} : \quad & T(\xi, y, z) = T(x, y, \xi) = 0, \\
 & T(x, y, z) = -T(\varphi x, \varphi y, z) = \mp T(x, \varphi y, \varphi z); \\
 \mathcal{T}_{13} : \quad & T(\xi, y, z) = T(x, y, \xi) = 0, \\
 & T(x, y, z) - T(\varphi x, \varphi y, z) = \underset{x, y, z}{\mathfrak{S}} T(x, y, z) = 0; \\
 \mathcal{T}_{14} : \quad & T(\xi, y, z) = T(x, y, \xi) = 0, \\
 & T(x, y, z) - T(\varphi x, \varphi y, z) = \underset{x, y, z}{\mathfrak{S}} T(\varphi x, y, z) = 0; \\
 \mathcal{T}_{21/22} : \quad & T(x, y, z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi) = \mp \eta(z)T(\varphi x, \varphi y, \xi); \\
 \mathcal{T}_{31/32} : \quad & T(x, y, z) = \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) - \eta(y)T(\xi, \varphi^2 x, \varphi^2 z), \\
 & T(\xi, y, z) = \pm T(\xi, z, y) = -T(\xi, \varphi y, \varphi z); \\
 \mathcal{T}_{33/34} : \quad & T(x, y, z) = \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) - \eta(y)T(\xi, \varphi^2 x, \varphi^2 z), \\
 & T(\xi, y, z) = \pm T(\xi, z, y) = T(\xi, \varphi y, \varphi z); \\
 \mathcal{T}_{41} : \quad & T(x, y, z) = \eta(z) \{ \eta(y)\hat{t}(x) - \eta(x)\hat{t}(y) \},
 \end{aligned}$$

where \mathfrak{S} stands for the cyclic sum by three arguments and the former and latter subscripts of $\mathcal{T}_{ij/kl}$ correspond to upper and down signs plus or minus, respectively.

According to [13], the φ B-connection (therefore, in our case, the φ -canonical connection) belongs to the following class

$$\mathcal{T}_{12} \oplus \mathcal{T}_{13} \oplus \mathcal{T}_{14} \oplus \mathcal{T}_{21} \oplus \mathcal{T}_{22} \oplus \mathcal{T}_{31} \oplus \mathcal{T}_{32} \oplus \mathcal{T}_{33} \oplus \mathcal{T}_{34} \oplus \mathcal{T}_{41}.$$

In [15], the basic classes of the almost contact B-metric manifolds are characterized by conditions for the torsion of the φ -canonical connection. For the classes under consideration we have:

$$\begin{aligned}
 \mathcal{F}_1 : \quad & T^0(x, y) = \frac{1}{2n} \{ t^0(\varphi^2 x)\varphi^2 y - t^0(\varphi^2 y)\varphi^2 x + t^0(\varphi x)\varphi y - t^0(\varphi y)\varphi x \}; \\
 \mathcal{F}_4 : \quad & T^0(x, y) = \frac{1}{2n} t^{0*}(\xi) \{ \eta(y)\varphi x - \eta(x)\varphi y \}; \\
 \mathcal{F}_5 : \quad & T^0(x, y) = \frac{1}{2n} t^0(\xi) \{ \eta(y)\varphi^2 x - \eta(x)\varphi^2 y \}; \\
 \mathcal{F}_{11} : \quad & T^0(x, y) = \{ \hat{t}^0(x)\eta(y) - \hat{t}^0(y)\eta(x) \} \xi.
 \end{aligned}$$

Moreover, it is given the following correspondence between the classes \mathcal{F}_i of M and the classes \mathcal{T}_{jk} of T^0 :

$$\begin{aligned}
 (1.10) \quad & M \in \mathcal{F}_1 \Leftrightarrow T^0 \in \mathcal{T}_{13}, t^0 \neq 0; \\
 & M \in \mathcal{F}_4 \Leftrightarrow T^0 \in \mathcal{T}_{31}, t^0 = 0, t^{0*} \neq 0; \\
 & M \in \mathcal{F}_5 \Leftrightarrow T^0 \in \mathcal{T}_{31}, t^0 \neq 0, t^{0*} = 0; \\
 & M \in \mathcal{F}_{11} \Leftrightarrow T^0 \in \mathcal{T}_{41}.
 \end{aligned}$$

2. NATURAL CONNECTIONS IN THE MAIN CLASSES

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact B-metric manifold belonging to the main classes $\mathcal{F}_1, \mathcal{F}_4, \mathcal{F}_5$ and \mathcal{F}_{11} . The goal in the present section is a determination of the general form of the torsion T of a natural connection D on $(M, \varphi, \xi, \eta, g)$.

Bearing in mind the form of the torsion of the φ -canonical connection (1.8) and characteristic conditions (1.3) of the mentioned classes, we expect that the torsion T is expressed by the components from (1.1) and (1.2) of the metrics g and \tilde{g} , respectively.

Let the torsion T of a linear connection D have the following form

$$\begin{aligned}
 (2.1) \quad T(x, y, z) = & g(\varphi y, \varphi z)\vartheta_1(x) + g(y, \varphi z)\vartheta_2(x) + \eta(y)\eta(z)\vartheta_3(x) \\
 & - g(\varphi x, \varphi z)\vartheta_1(y) - g(x, \varphi z)\vartheta_2(y) - \eta(x)\eta(z)\vartheta_3(y),
 \end{aligned}$$

where, for $\lambda_i \in \mathbb{R}$ ($i = 1, 2, \dots, 18$), there are denoted the following 1-forms:

$$\begin{aligned}
 (2.2) \quad \vartheta_1(x) = & \lambda_1\theta(\varphi^2x) + \lambda_3\theta(\xi)\eta(x) + \lambda_5\omega(x) \\
 & + \lambda_2\theta^*(\varphi^2x) + \lambda_4\theta^*(\xi)\eta(x) + \lambda_6\omega(\varphi x), \\
 \vartheta_2(x) = & \lambda_7\theta(\varphi^2x) + \lambda_9\theta(\xi)\eta(x) + \lambda_{11}\omega(x) \\
 & + \lambda_8\theta^*(\varphi^2x) + \lambda_{10}\theta^*(\xi)\eta(x) + \lambda_{12}\omega(\varphi x), \\
 \vartheta_3(x) = & \lambda_{13}\theta(\varphi^2x) + \lambda_{15}\theta(\xi)\eta(x) + \lambda_{17}\omega(x) \\
 & + \lambda_{14}\theta^*(\varphi^2x) + \lambda_{16}\theta^*(\xi)\eta(x) + \lambda_{18}\omega(\varphi x).
 \end{aligned}$$

We set the condition D to be a natural connection on $(M, \varphi, \xi, \eta, g)$. According to Proposition 1.1 and (1.5), we obtain the following relation

$$\begin{aligned}
 (2.3) \quad F(x, y, z) = & \frac{1}{2} \{ T(x, y, \varphi z) - T(y, \varphi z, x) + T(\varphi z, x, y) \\
 & - T(x, \varphi y, z) + T(\varphi y, z, x) - T(z, x, \varphi y) \}.
 \end{aligned}$$

Then, applying (2.1) and (2.2) to (2.3) and comparing with (1.4), we obtain the following conditions for the parameters:

$$\lambda_1 + \lambda_8 = \lambda_5 - \lambda_{12} = \lambda_6 + \lambda_{11} = 0, \quad \lambda_2 - \lambda_7 = -\lambda_4 = \lambda_9 = \frac{1}{2n},$$

$$\lambda_3 = \lambda_{10} = \lambda_{13} = \lambda_{14} = \lambda_{15} = \lambda_{16} = \lambda_{17} = 0, \quad \lambda_{18} = -1.$$

In consequence, the 1-forms from (2.2) take the form

$$\begin{aligned} \vartheta_1(x) &= \lambda_1 \theta(\varphi^2 x) + \lambda_2 \theta^*(\varphi^2 x) - \frac{1}{2n} \theta^*(\xi) \eta(x) \\ &\quad + \lambda_5 \omega(x) + \lambda_6 \omega(\varphi x), \\ (2.4) \quad \vartheta_2(x) &= \left(\lambda_2 - \frac{1}{2n} \right) \theta(\varphi^2 x) - \lambda_1 \theta^*(\varphi^2 x) + \frac{1}{2n} \theta(\xi) \eta(x) \\ &\quad - \lambda_6 \omega(x) + \lambda_5 \omega(\varphi x), \\ \vartheta_3(x) &= -\omega(\varphi x). \end{aligned}$$

Further, we use the following tensors derived by the structural tensors of $(M, \varphi, \xi, \eta, g)$:

$$\begin{aligned} \pi_1(x, y)z &= \{g(y, z)x\}_{[x \leftrightarrow y]}, \quad \pi_2(x, y)z = \{g(y, \varphi z)\varphi x\}_{[x \leftrightarrow y]}, \\ \pi_3(x, y)z &= -\{g(y, z)\varphi x + g(y, \varphi z)x\}_{[x \leftrightarrow y]}, \\ \pi_4(x, y)z &= \{\eta(y)\eta(z)x + g(y, z)\eta(x)\xi\}_{[x \leftrightarrow y]}, \\ \pi_5(x, y)z &= \{\eta(y)\eta(z)\varphi x + g(y, \varphi z)\eta(x)\xi\}_{[x \leftrightarrow y]}, \end{aligned}$$

as well as the corresponding tensors $\pi_i(x, y, z, w) = g(\pi_i(x, y)z, w)$, $i = 1, \dots, 5$, of type (0,4). Let us note that the latter tensors are curvature-like tensors, i.e. they have the properties of the curvature tensor $R = [\nabla, \nabla] - \nabla_{[\cdot]}$ of type (0,4).

In (2.4), we rename the parameters as follows $\alpha_1 = \lambda_1$, $\alpha_2 = \lambda_2$, $\alpha_3 = \lambda_5$, $\alpha_4 = \lambda_6$. Then, using (2.1) and (2.4), we establish the truthfulness of the following

Theorem 2.1. *The torsion of any natural connection on an almost contact B-metric manifold in $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$ is expressed by*

$$\begin{aligned} (2.5) \quad T(x, y, z) &= (\pi_3 + \pi_5)(x, y, z, q) \\ &\quad + \frac{1}{2n}(\pi_2 + \pi_4)(x, y, z, a^*) + \frac{1}{2n}\pi_5(x, y, z, a) - \pi_5(x, y, z, \hat{a}), \end{aligned}$$

where $q = \alpha_1 \varphi^2 a^* - \alpha_2 \varphi^2 a - \alpha_3 \varphi \hat{a} + \alpha_4 \hat{a}$ ($\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$) and $\theta = g(\cdot, a)$, $\theta^* = g(\cdot, a^*)$, $\omega = g(\cdot, \hat{a})$.

Actually, the tensors in (2.5) form a 4-parametric family.

By direct check, we establish that equality (2.5) implies the following

Proposition 2.2. *The torsion of any natural connection on an almost contact B-metric manifold in $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$ has the property*

$$\bigotimes_{x,y,z} T(x, y, z) = 0.$$

As a corollary, using (1.5) and Proposition 2.2, we obtain immediately $Q(x, y, z) = T(z, y, x)$, i.e. we have the following

Proposition 2.3. *Any natural connection D with torsion T on an almost contact B-metric manifold from $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$ is determined by T and the Levi-Civita connection ∇ as follows:*

$$g(D_x y, z) = g(\nabla_x y, z) + T(z, y, x).$$

3. SPECIAL NATURAL CONNECTIONS IN THE MAIN CLASSES

Bearing in mind that the φ -canonical connection (resp., the φ B-connection) is a natural connection on $(M, \varphi, \xi, \eta, g)$, then we have to determine its corresponding parameters α_i ($i = 1, 2, 3, 4$) in (2.5).

Theorem 3.1. *The natural connection with torsion T from the family (2.5) is the φ -canonical connection on a $(2n+1)$ -dimensional almost contact B-metric manifold from $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$ if and only if $\alpha_1 = \alpha_3 = \alpha_4 = 0$, $\alpha_2 = \frac{1}{4n}$. Then its torsion has the form*

$$(3.1) \quad \begin{aligned} T^0(x, y, z) &= \frac{1}{4n}(\pi_1 + \pi_2 + \pi_4)(x, y, z, a^*) \\ &+ \frac{1}{2n}\pi_5(x, y, z, a) - \pi_5(x, y, z, \hat{a}). \end{aligned}$$

Proof. The statement follows immediately from (2.5), (1.8) and (1.4). \square

By virtue of (3.1) and (1.9), we obtain the following

Proposition 3.2. *The φ -canonical connection on an almost contact B-metric manifold from $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$ has a torsion belonging to the class $\mathcal{T}_{13} \oplus \mathcal{T}_{31} \oplus \mathcal{T}_{41}$.*

The latter fact confirms the results in (1.10).

Theorem 2.1 gives an one-to-one map of the set of the natural connections onto the set of the quadruples $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$. According to Theorem 3.1, the φ -canonical connection is determined by $(0, \frac{1}{4n}, 0, 0)$. The interesting case is the connection from the family (2.5) for $(0, 0, 0, 0)$. This natural connection we call the *standard connection* on $(M, \varphi, \xi, \eta, g) \in \mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$.

Then, according to (2.5), its torsion has the form

$$(3.2) \quad \begin{aligned} T'(x, y, z) &= \frac{1}{2n}(\pi_2 + \pi_4)(x, y, z, a^*) \\ &+ \frac{1}{2n}\pi_5(x, y, z, a) - \pi_5(x, y, z, \hat{a}). \end{aligned}$$

Using (3.2) and (1.9), we get the following

Proposition 3.3. *The standard connection on an almost contact B-metric manifold from $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$ has a torsion belonging to the class $\mathcal{T}_{11} \oplus \mathcal{T}_{13} \oplus \mathcal{T}_{31} \oplus \mathcal{T}_{41}$.*

There exist a natural connection ∇'' with torsion T'' from the family (2.5) of a remarkable role. This connection is determined by the condition the φ -canonical connection to be the average connection of ∇' and ∇'' . Therefore we have

$$(3.3) \quad \begin{aligned} T''(x, y, z) &= \frac{1}{2n}(\pi_3 + \pi_5)(x, y, z, a) + \frac{1}{2n}(\pi_2 + \pi_4)(x, y, z, a^*) \\ &+ \frac{1}{2n}\pi_5(x, y, z, a) - \pi_5(x, y, z, \hat{a}). \end{aligned}$$

Bearing in mind (3.3) and (1.9), we get the following

Proposition 3.4. *The natural connection ∇'' on an almost contact B-metric manifold from $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$ has a torsion belonging to the class $\mathcal{T}_{11} \oplus \mathcal{T}_{13} \oplus \mathcal{T}_{31} \oplus \mathcal{T}_{41}$.*

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